Countering Indifference Using Counterintuitive Examples

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Summary
Counterintuitive examples can motivate not demoralise students.

◆INTRODUCTION◆

THERE has been more confusion than consensus concerning the role of counterintuitive examples in statistics curriculum and pedagogy. It is the author's belief that wise use of the rich natural supply of counterintuitive examples in statistics supports the constructivist pedagogy being called for in educational reform by acknowledging students' prior beliefs, engaging those beliefs to yield deeper understanding, and supporting the role of teacher as facilitator of those "what if" questions. Additional benefits to students include opportunities for: motivation, metacognition, critical thinking, discovery learning, connections to real-life applications and history. This article will present a synthesis between perspectives, and illustrate it with a specific counterintuitive example, Simpson's Paradox.

◆A PAIR OF PARADIGMS◆ ON PARADOX

Counterintuitive situations in probability and statistics abound in real life. The fall 1994 issue of Chance magazine had three articles involving such situations, including the prosecutor's fallacy of confusing P(A|B) with P(B|A), and a paradox in which relative rankings of skaters who have finished depend on participants yet to skate. Other "famous" examples include the Birthday Problem, Simpson's Paradox and the Monty Hall "three doors" problem. Despite the common occurrence of counterintuitive examples, there is little guidance on their place in the curriculum.

The traditional view is perhaps best represented by Burrill (1990, p. 113), the current president of the National Council of Teachers of Mathematics, and is repeated by the American Statistical Association (1994):

"The emphasis in teaching statistics should be on good examples and building intuition, not on probability paradoxes or using statistics to deceive." Supporting this view is the common concern expressed by Falk and Konold (1992, p. 161): "It is tempting to bring some of the more devious problems to the classroom to demonstrate to students their erroneous tendencies and perhaps enlighten them. However, if a teacher persists in pointing out to students how prone they are to inferential errors, they may become so convinced of their incapacities that they despair of ever mastering more appropriate techniques."

An alternative view is perhaps best expressed by Gordon (1991, p. 511): "Not only do [counterintuitivel instances...gain students' attention...but such examples also help students challenge habits of thought and practices, thus leading to their becoming better thinkers.... [gaining] a greater appreciation of the need for exploration, reflection and reasoning." The author found (Lesser 1995) that active group discussions naturally generated by such examples are supported by current curriculum reform movements and by several specific pedagogical approaches such as structured controversy, critical thinking, conflict teaching and errors as a springboard to inquiry.

It is the author's belief that there is a middle ground in which the wise use of the rich natural supply of counterintuitive examples in statistics (many of which are catalogued in Lesser 1995) supports the pedagogy of constructivism being called for in educational reform by acknowledging students' prior beliefs, engaging those beliefs to yield deeper understanding, and supporting the role of teacher as facilitator of those "what if" questions. The essence of constructivism (e.g. Lerman 1989) is that students must actively construct their knowledge, rather than passively receive it from the environment. What better way to force such an active construction than with an example sufficiently rich in texture and context that it simply cannot be gulped down without rumination.

◆A SPECIFIC◆ COUNTERINTUITIVE EXAMPLE: SIMPSON'S PARADOX

Simpson's Paradox, first noted in 1951 by British statistician E. H. Simpson, occurs when a comparison is reversed under aggregation. In the simplified example data below, students can verify for themselves that women were hired at a higher rate than men within each department, yet were hired at a lower rate for the entire university.

	WOMEN		MEN	
	Hired	Applied	Hired	Applied
HUMANITIES	30	80	5	20
SCIENCES	15	20	50	80

When shown this example among more conventional weighted mean problems, a student commented: "I like that one [Simpson's Paradox) better even though I didn't understand it at first." This suggests that students who are shown something with a counterintuitive twist in a real-life context may be more likely to want to stick with it and "figure it out". Students at our school engaged in a lively discussion of a hypothetical court case about these data with some students speaking as a lawyer for the university ("we hire women at least as much as men in both departments"), some speaking as a women's rights lawyer ("women are still hired at a lower rate than men at the university"), and others trying to reconcile the true statements of both parties with a deeper understanding of the structure of the overall situation ("what's the role of the sample sizes?").

While the numbers were chosen to be analysed easily by hand, this example is by no means hypothetical. Simpson's Paradox has actually occurred in several real-life situations (e.g., Cohen 1986), including: university admission rates, rural/urban fertility rates, young/old morbidity rates, categories of federal tax rates, various baseball statistics, and death penalty cases by race. Again, there is little consensus on if or how to present this. Many standard introductory statistics textbooks avoid it altogether, while the popular Moore and McCabe (1993) book uses it in every three-way table example! Lesser (1995) illustrates this specific example in several ways, including a geometric model, a platform scale and a circle graph. The platform scale model is shown in Figure 1. More recently, Bea has added the unit square model (e.g., Bea and Scholz 1994) representation. Students with a solid mathematics background can find additional representations in Lord (1990): arguments of complex numbers, linear transformations of the plane and determinants of matrices.

This particular counterintuitive example can be used to confront common student confusion between weighted and unweighted means (the averaging-the-averages misconception). It also forces an awareness of what one is averaging over, such as the phenomenon that a university's mean class size averaged over students is never smaller than the mean class size averaged over classes. For example, in a university with a class of 90 and 10, the average class size appears to be (90 + 10)/2

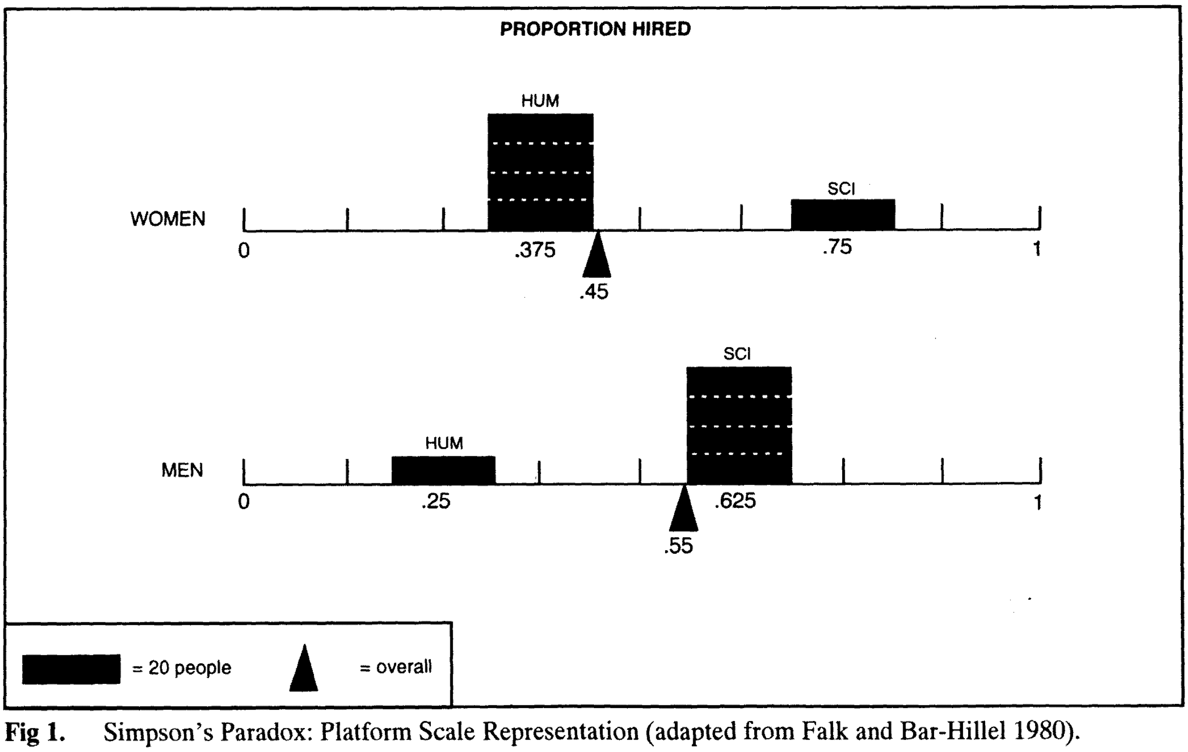

Fig 1. Simpson's Paradox: Platform Scale Representation (adapted from Falk and Bar-Hillel 1980).

= 50, but per student it is [(90)(90)+ (10)(10) 1/100 = 82.

◆SURVEY◆

On the first day of class during the Spring 1997 semester, a total of 97 students who were enrolled in three sections of the service introductory statistics course at a mid-sized state-supported university in the Rocky Mountain region took the author's Statistics Opinion Survey. This course requires only a "C" in high school algebra, and so the author believes these results should be comparable with a high school statistics course. The SOS (which had a Cronbach's alpha reliability coefficient of .84) included 20 self-contained true statements in layperson's language which students were asked to rate from 1 to 5 according to both their interest in the statement and the degree to which they found the result surprising (i.e., counterintuitive). The SOS included "known" counterintuitive statements (e.g. the first two sample statements) as well as "straight" statements (see last sample statement). Students were directed not to try calculations, but to "just use your first impression of the statement."

◆SAMPLE STATEMENTS◆ FROM SOS SURVEY

The items from the survey sampled several distinct topics such as independence, combinatorics, regression to the mean, sampling, conditional probability and (weighted and unweighted) means. Because this survey was given before instruction began, the statements are written with informal language that does not always have full mathematical or technical precision.

- If each department hires women at a higher rate than it hires men, it is still possible that the overall college hiring rate for women is less than the overall rate for men.
- The average class size at a university is always bigger from the student's point of view than it is from the teacher's point of view.
- When picking a sample that's a small fraction of a large population, it doesn't really matter whether the same person could get chosen more than once.

◆RESULTS AND DISCUSSION◆

The overall correlation between interest and surprise was 0.666, which is significant at the .001 level. The statements which were more counterintuitive tended to generate the most interest from the students, a critical association for a subject students too often do not find interesting enough. Interestingly enough, two of the three items ranked most interesting were the "Simpson's paradox" statement and the related "weighted averages" statement (the first two sample statements listed).
Instructors interested in including more counterintuitive examples in their curriculum should consult Lesser (1995), which not only catalogues many of them in a sequence linked to the "traditional" order of topics presentation, but provides a framework for their selective use. A primary recommendation of this framework is to limit examples to those that actually occur in real life (this eliminates contrived probability paradoxes, but still leaves plenty of examples to choose from) and that can be readily explained or explored by means other than analytic mathematics alone (examples of this latter criterion were given for Simpson's Paradox).

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